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OPTIMUM STRESS DESIGN OF A ROTATING WIRE ANTENNA

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SUMMARY

This memorandum investigates the problem of designing a rotating wire antenna which might be extended from a satellite vehicle to act as an intermittent reflector of radio signals between sending and receiving stations on the ground. An optimum variation of the cross section of such a wire is determined so as to maximize its length without exceeding the tensile strength of the material or reducing the cross section below a specified minimum value.

Numerical results indicate that for rotation rates of the order of 1 radian per second, the allowable lengths are greater than 12,000 ft for both steel and metallically coated glass fibers.

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SYMBOLS

$A(x)$	Cross-sectional area of the wire as a function of distance from axis of rotation
A_o	Maximum allowable cross-sectional area
A_1	Minimum allowable cross-sectional area
a	Total length of extended wire
a_o	Total length of the variable-area segment of the wire
$T(x)$	Tension force as a function of distance from axis of rotation
V_p	Maximum allowable peripheral velocity at the end of the wire
x	Distance from axis of rotation along the wire
$\dot{\alpha}$	Actual rotation rate
$\dot{\alpha}_o$	Design rotation rate
ξ	Variable of integration
ρ_o	Density of the wire
$\tau(x)$	Tensile stress as a function of distance from axis of rotation
τ_o	Tensile strength of the wire

I. INTRODUCTION

It has recently been proposed that a communication satellite might consist of a long wire antenna rotating in the satellite orbital plane and maintained in a straight line by centrifugal force. Such an antenna could provide intermittent communication between a sending and a receiving station in much the same manner as in a meteor scatter communication system.

A number of mechanical problems arise in connection with such an antenna system. For instance, there may be difficulties in unreeling the antenna from a satellite and establishing the desired rotation rate without producing undesirable oscillatory transients relative to the required straight-line configuration. However, the present memorandum is concerned with the problem of specifying the design of an antenna that will maximize the length for a given material and rotation rate without exceeding the tensile strength of the material.

II. ANALYSIS

STATEMENT OF THE PROBLEM

The problem of designing a rotating wire antenna reduces to that of determining a functional relation between cross-sectional area and distance from the axis of rotation, so that the length of the wire is maximized without exceeding the tensile strength of the material at any point. Actually, this solution corresponds to the best-possible approximation to a uniform stress solution in which the tensile stress is equal to the tensile strength at every point along the wire. In obtaining such a solution, it is desirable to specify a maximum and a minimum allowable cross-sectional area, A_o and A_1, respectively. The value of A_o would be determined on the basis of the size of wire which could be conveniently wound on a reel in a satellite vehicle, while the value of A_1 would depend on how fine a wire could be handled.

METHOD OF SOLUTION

Uniform Tensile Stress Solution

In order to maximize the length of the wire for a given material and design rotation rate, it is desirable to vary the cross-sectional area in such a way that the tensile stress (tension per unit area) is constant along the length of the wire.

Figure 1 represents the unknown variation of area, $A(x)$, as a function of the distance, x, from the axis of rotation, and ξ is a running variable of integration. If ρ_o is the density of the material and $\dot{\alpha}_o$ is the design angular rate, then the tension force $T(x)$ can be expressed by

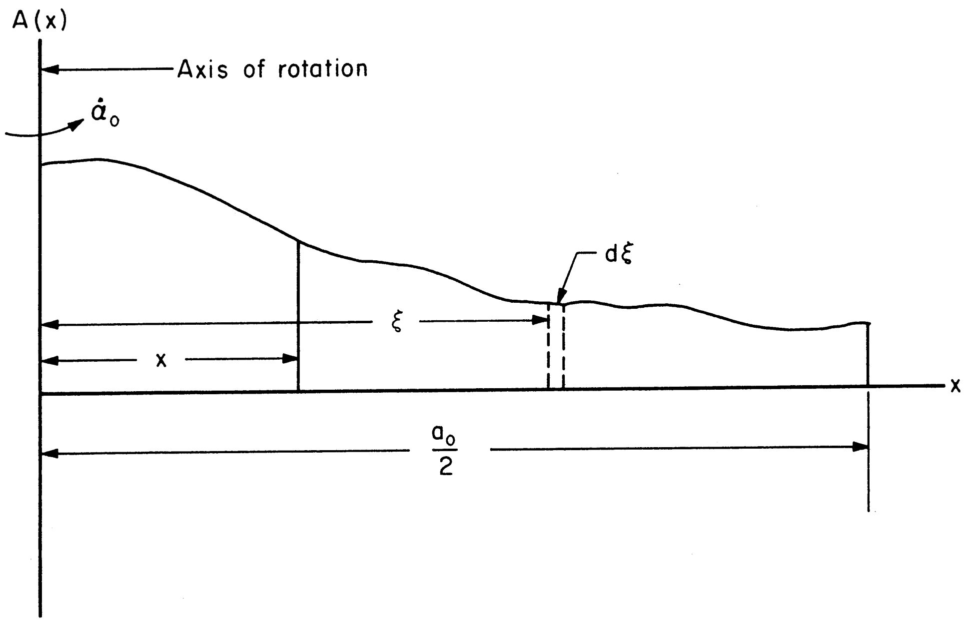

Fig. 1— Geometry of the problem

$$T(x) = \rho_o \dot{\alpha}_o^2 \int_x^{\frac{a_o}{2}} \xi \; A(\xi) \, d\xi \tag{1}$$

where $a_o/2$ is the distance from the axis of rotation to the end of the wire.

If τ_o represents the tensile strength of the material, then the condition for constant tensile stress is given by

$$T(x) = \tau_o A(x) \tag{2}$$

Thus, by combining Eqs. (1) and (2), an integral equation for the area variation is obtained:

$$A(x) = \frac{\rho_o \dot{\alpha}_o^2}{\tau_o} \int_x^{\frac{a_o}{2}} \xi \; A(\xi) \, d\xi \tag{3}$$

This equation has the form of a Volterra integral equation which has the property that for a finite value of $a_o/2$ the only solution that satisfies both Eqs. (1) and (2) is given by

$$A(x) \equiv 0 \tag{4}$$

While this solution is mathematically correct, the problem of fabricating a wire of length $a_o/2$ and zero cross-sectional area seems to be rather formidable.

On the other hand, if the length $a_o/2$ is infinite, then the solution of Eq. (3) does satisfy both Eqs. (1) and (2) and is of the form

$$A(x) = A_o \, e^{-\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} x^2} \tag{5}$$

where A_o is the cross-sectional area at the axis of rotation where $x = 0$. While a wire of this design rotating at a rate $\dot{\alpha}_o$ would have a constant

value of tensile stress τ_o at any point, the construction of a variable-area wire of infinite length is still beyond the current state of the art.

Thus it appears that no exact constant stress solution exists for a wire of finite length.

Finite Wire Stress Solution

It is of interest to see how closely a constant stress solution is approached if an area variation corresponding to Eq. (5) for the infinite wire is assumed from x = 0 to $a_o/2$. Thus

$$A(x) = A_o e^{-\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} x^2} \qquad \left(0 < x < \frac{a_o}{2}\right) \tag{6}$$

$$A(x) = 0 \qquad \left(x > \frac{a_o}{2}\right) \tag{7}$$

At this point, it is convenient to introduce the cross-sectional area, A_1, at the end of the wire where $x = \frac{a_o}{2}$. Thus from Eq. (6)

$$A_1 = A_o e^{-\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} \left(\frac{a_o}{2}\right)^2} \tag{8}$$

which can be solved for $a_o/2$ in terms of A_1 and A_o:

$$\frac{a_o}{2} = \left[\frac{2\tau_o}{\rho_o \dot{\alpha}_o^2} \log \frac{A_1}{A_o}\right]^{1/2} \tag{9}$$

Equation (9) then gives the length of wire as a function of the material (ρ_o, τ_o) and the ratio of the terminal to the initial area, A_1/A_o.

Since the function A(x) is monotonic, A_o and A_1 can be regarded as the maximum and minimum permissible areas, respectively. In practice A_o is

limited by the diameter of wire which can be wound on a reel in the satellite vehicle without undergoing a permanent set, while A_1 is limited by the smallest-size wire which can be handled conveniently.

The tension force along such a wire can be expressed by the relation

$$T(x) = \rho_o \dot{\alpha}^2 \int_o^{\frac{a_o}{2}} \xi \quad A(\xi) \quad d\xi \tag{10}$$

where $\dot{\alpha}$ is the actual rotation rate rather than the design rotation rate $\dot{\alpha}_o$ used in Eq. (1), and $A(\xi)$ is given by Eq. (6).

Integration of Eq. (10) gives the following:

$$T(x) = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \quad \tau_o A_o \left[e^{-\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} x^2} - e^{-\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} \left(\frac{a_o}{2}\right)^2} \right] \tag{11}$$

Division of Eq. (11) by Eq. (6) then gives the tensile stress as

$$\tau(x) = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \quad \tau_o \left[1 - e^{-\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} \left(\frac{a_o^2}{4} - x^2\right)} \right] \tag{12}$$

which can be further simplified by means of Eq. (8) to give

$$\tau(x) = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \quad \tau_o \left[1 - \left(\frac{A_1}{A_o}\right)^{1 - \frac{4x^2}{a_o^2}} \right] \tag{13}$$

Figure 2 is a non-dimensional plot of Eq. (13) for $\dot{\alpha} = \dot{\alpha}_o$, giving the ratio of tensile stress to tensile strength, τ/τ_o, as a function of the ratio of the distance from the axis of rotation to the total length, $2x/a_o$, for

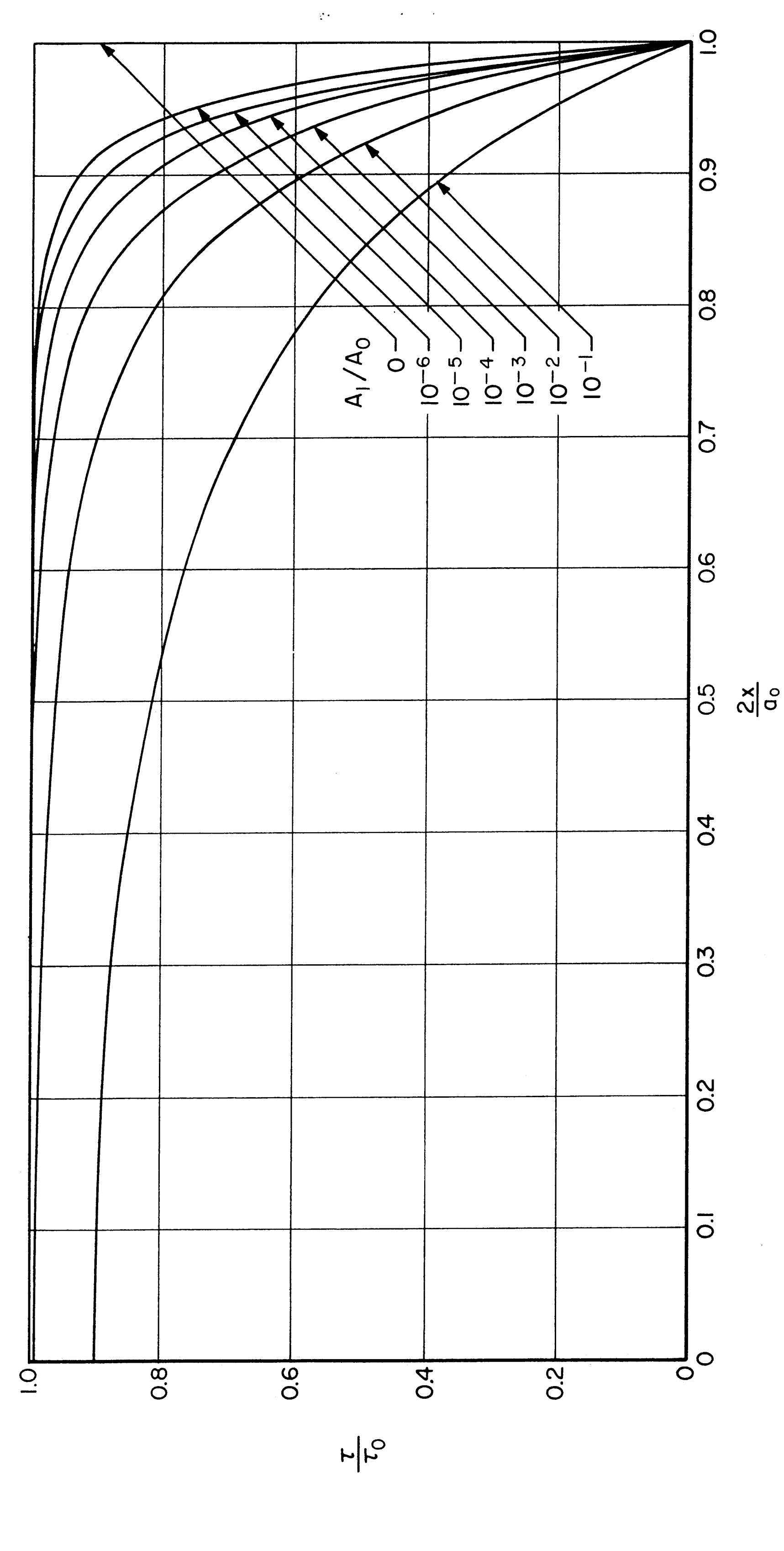

Fig. 2 — Tensile stress in a finite wire

values of A_1/A_o equal to 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, and 10^{-6}. An examination of Fig. 2 shows that as the area ratio A_1/A_o becomes smaller, the corresponding stress solution approaches the uniform case of $\tau = \tau_o$, but in all cases the stress drops to zero at $x = \frac{a_o}{2}$, as would be expected. Also, the factor $\dot{\alpha}^2/\dot{\alpha}_o^2$ in Eq. (13) shows that as long as $\dot{\alpha} < \dot{\alpha}_o$ the stress, τ, will not exceed the tensile strength, τ_o, at any point.

Figures 3 and 4 are plots of Eq. (9) giving the length of the wire, $a_o/2$, as a function of the design angular rate, $\dot{\alpha}_o$, for steel and glass, respectively, for the same values of A_1/A_o used in Fig. 2. A comparison of Figs. 3 and 4 shows that glass is superior to steel because the ratio of tensile strength to density, τ_o/ρ_o, is larger. Obviously, the glass would have to have a metallic coating for electrical reasons, but the glass would still supply the strength.

Extension at Uniform Area

An examination of Fig. 2 shows that the tensile strength of the material is not being used efficiently in the design specified above, particularly at the outer end of the wire. In view of this, it appears that the wire could be lengthened somewhat without exceeding the tensile strength or reducing the cross-sectional area below the minimum value A_1.

The best method of maximizing the wire's length is to add to the wire described above, a cylindrical extension of cross-sectional area A_1 and of a length such that the resulting tension force at $a_o/2$ is the same as if the wire had had an infinite length and a cross section given by Eq. (5). This condition is specified by the relation

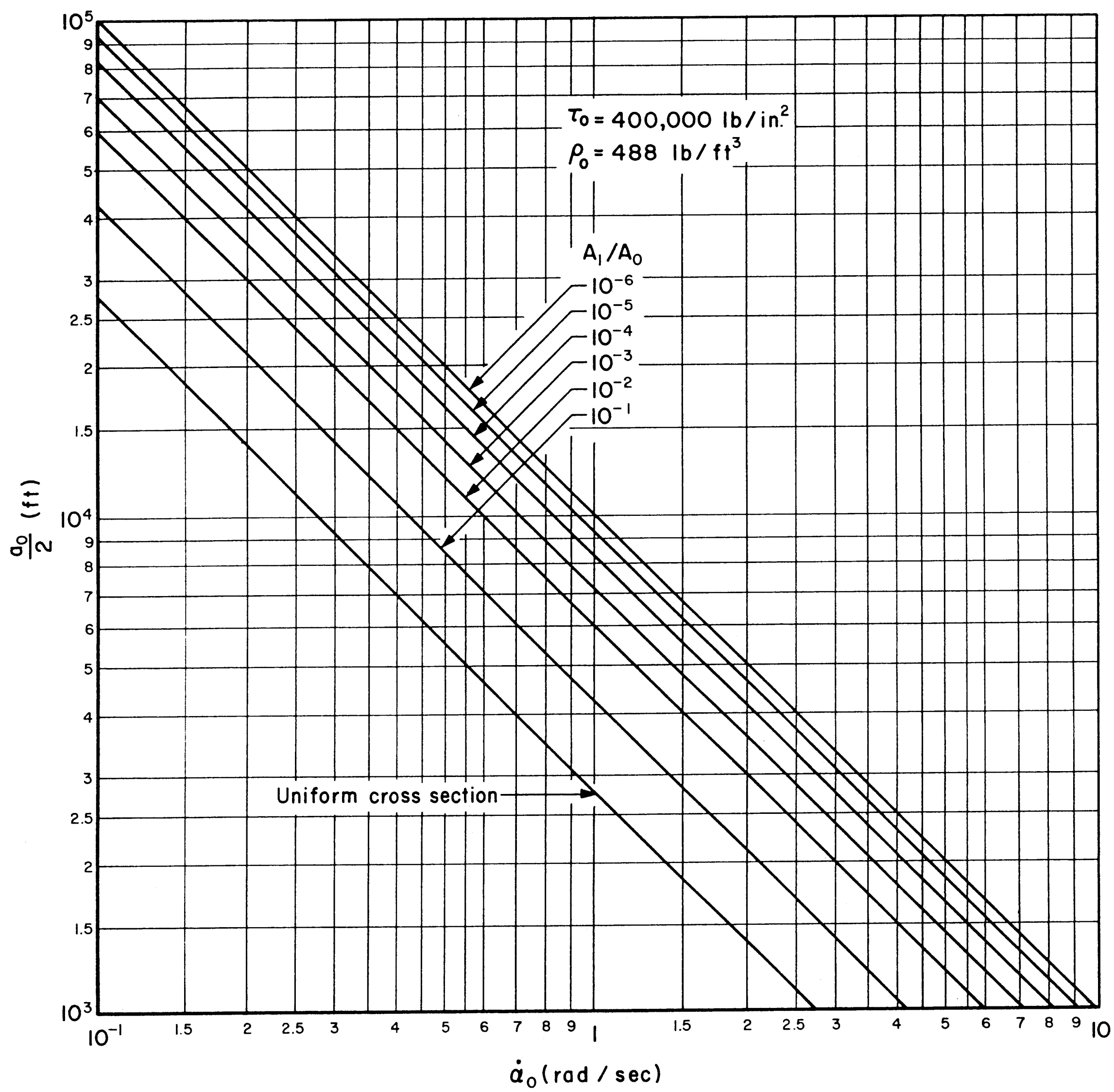

Fig. 3 — Length of wire vs angular rate

(Steel)

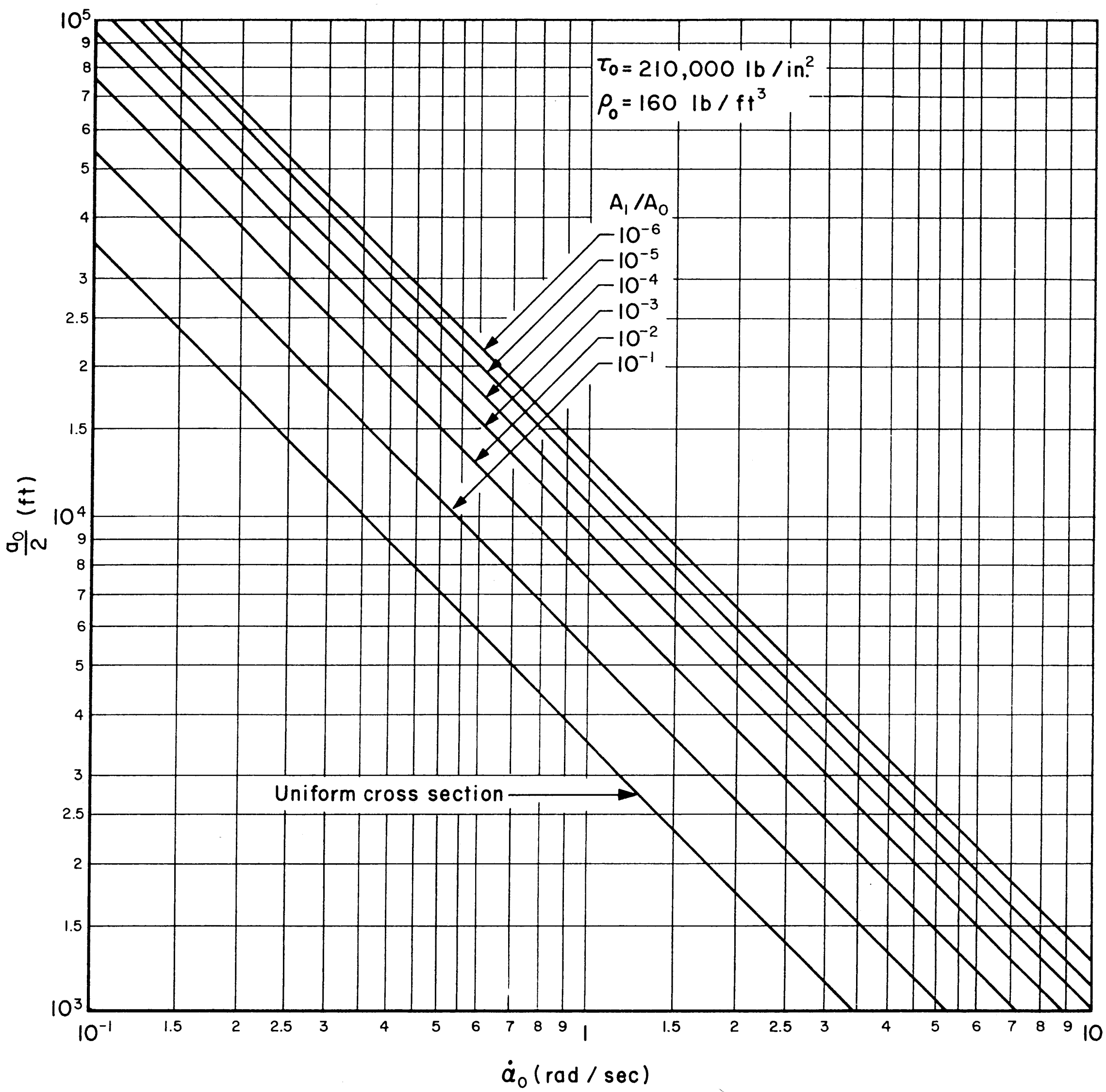

Fig. 4— Length of wire vs angular rate

(Glass)

$$\rho_o \dot{\alpha}_o^2 \int_{\frac{a_o}{2}}^{\frac{a}{2}} \xi A_1 \, d\xi = \rho_o \dot{\alpha}_o^2 A_o \int_{\frac{a_o}{2}}^{\infty} \xi e^{\frac{\rho_o \dot{\alpha}_o^2}{2\tau_o} \xi^2} d\xi \qquad (14)$$

where $a/2$ is the total length of the wire including the cylindrical extension.

By integrating Eq. (14) the following expression for $a/2$ is obtained:

$$\frac{a}{2} = \left[\left(\frac{a_o}{2} \right)^2 + \frac{2\tau_o}{\rho_o \dot{\alpha}_o^2} \right]^{1/2} \qquad (15)$$

Substitution of Eqs. (8) and (9) in Eq. (15) gives

$$\frac{a}{2} = \left[\frac{2\tau_o}{\rho_o \dot{\alpha}_o^2} \left(1 + \log \frac{A_o}{A_1} \right) \right]^{1/2} \qquad (16)$$

The tensile stress in the extended wire is determined by the fact that the cylindrical extension has the same effect as the infinite extension for the part of the wire out to $a_o/2$. Thus the tensile stress is given by

$$\tau(x) = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \tau_o \qquad (17)$$

for

$$0 < x < \frac{a_o}{2}$$

For the cylindrical extension beyond $a_o/2$ the tensile stress is given by

$$\tau(x) = \rho_o \dot{\alpha}^2 \int_{x}^{\frac{a}{2}} \xi \, d\xi \qquad (18)$$

which gives

$$\tau(x) = \frac{\rho_o \dot{\alpha}^2}{2} \left[\left(\frac{a}{2}\right)^2 - x^2 \right] \tag{19}$$

At $x = \frac{a_o}{2}$, Eqs. (17) and (19) should give the same value; thus

$$\frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \tau_o = \frac{\rho_o \dot{\alpha}^2}{2} \left[\left(\frac{a}{2}\right)^2 - \left(\frac{a_o}{2}\right)^2 \right] \tag{20}$$

Division of Eq. (19) by Eq. (20) gives

$$\tau(x) = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \tau_o \left[\frac{\left(\frac{a}{2}\right)^2 - x^2}{\left(\frac{a}{2}\right)^2 - \left(\frac{a_o}{2}\right)^2} \right] \tag{21}$$

Substitution of Eqs. (9) and (16) in Eq. (21) gives

$$\tau(x) = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \tau_o \left[1 + \left(1 - \frac{4x^2}{a_o^2}\right) \log \frac{A_o}{A_1} \right] \tag{22}$$

for

$$\frac{a_o}{2} < x < \frac{a_o}{2} \left[1 + \frac{1}{\log \frac{A_o}{A_1}} \right]^{1/2}$$

Equations (17) and (22) give the solution for the tensile stress variation along the extended wire.

Figure 5 shows a representative plot of the variation in cross-sectional area for a value of A_1/A_o of 0.1 as specified by Eqs. (6), (9), and (16). The quantity A/A_o is plotted against $\frac{2x}{a_o}$, giving a plot which is independent of the particular material used.

Figure 6 is a plot of Eqs. (17) and (22) giving the tensile stress as a function of position. As in Fig. 2, this plot is a non-dimensional form where τ/τ_o is plotted as a function of $2x/a_o$ for the same values of A_1/A_o

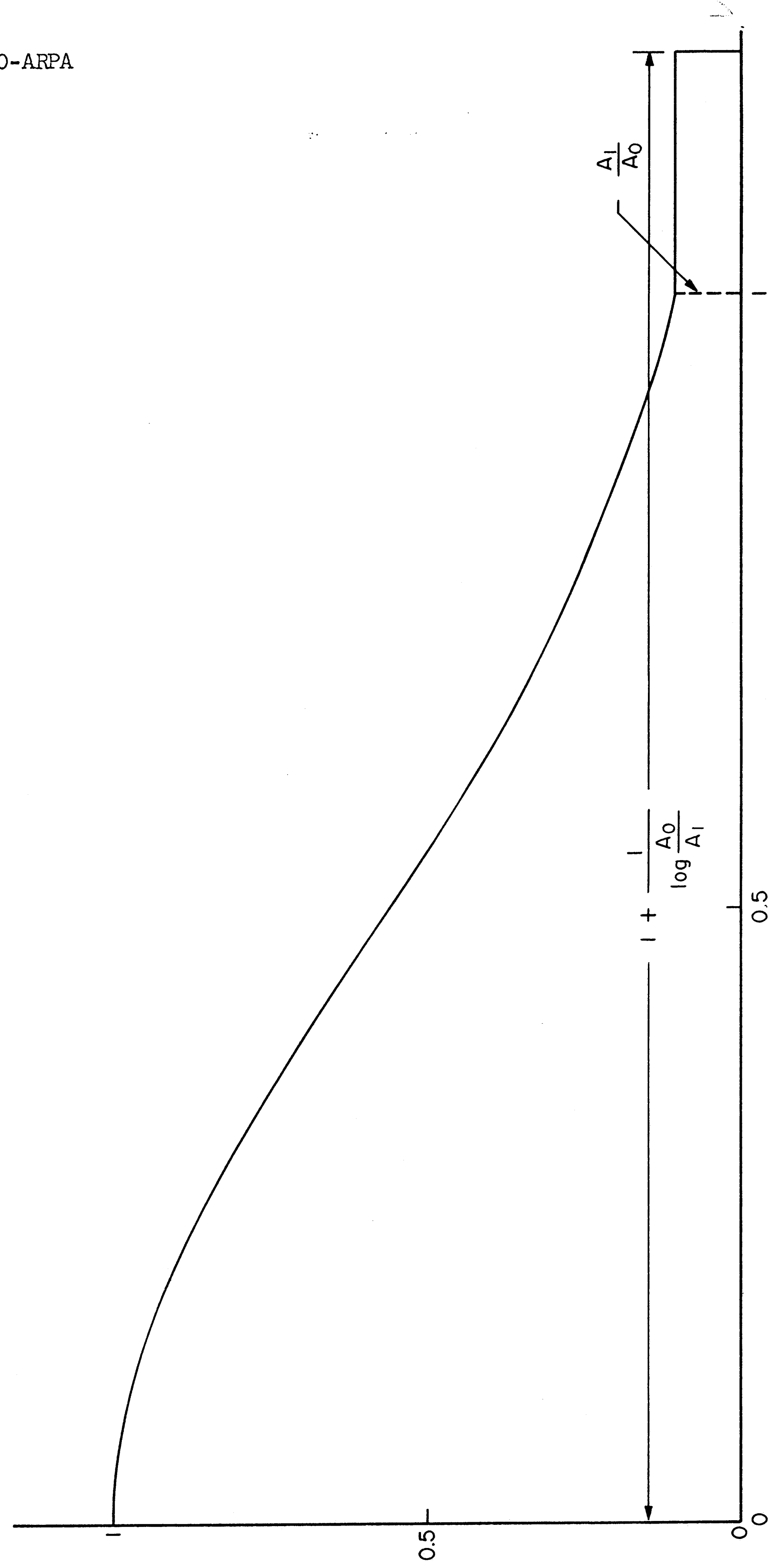

Fig. 5 — Optimum cross-section variation for extended wire

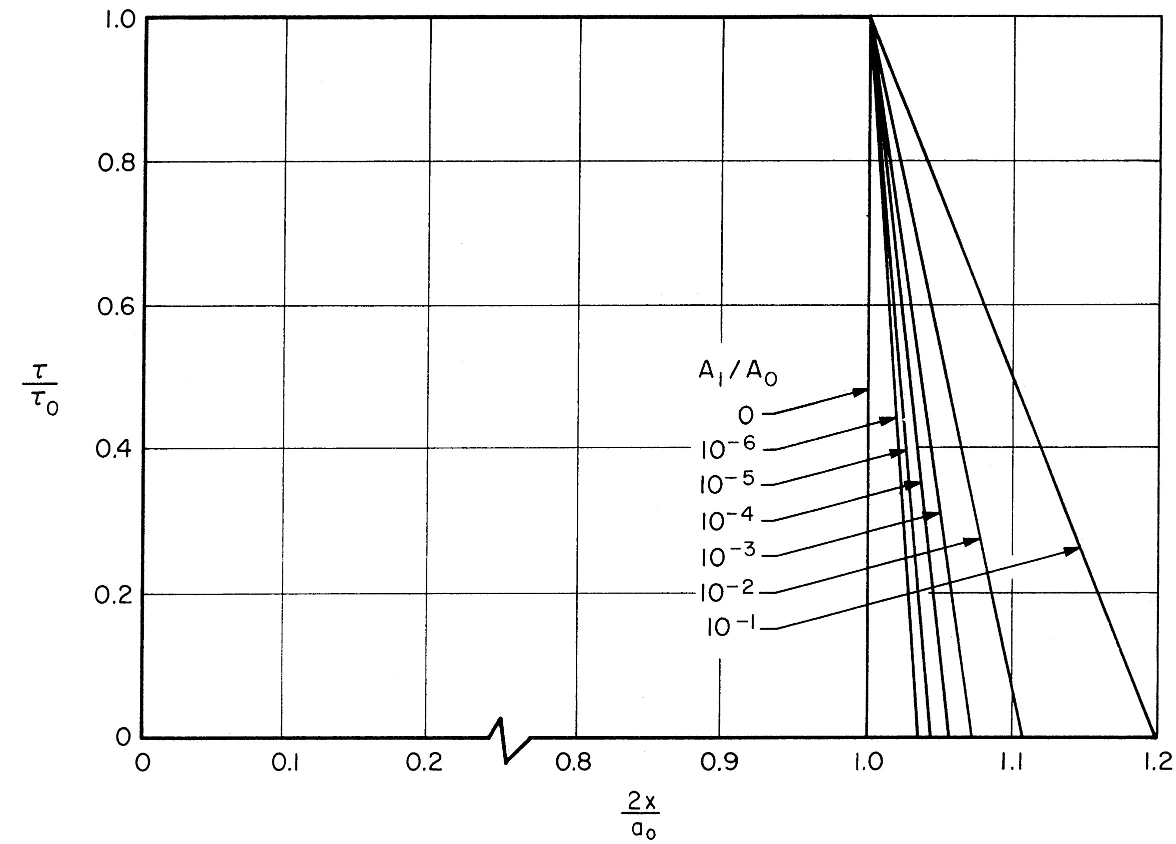

Fig. 6 — Tensile stress in an extended wire

as were used previously (10^{-1} to 10^{-6}).

A comparison of Figs. 2 and 6 shows that in the extended case the tensile stress is constant out to $x = a_o/2$, and then tapers off to zero in the cylindrical section. It is also seen that the percentage increase in length by the cylindrical extension decreases as the value of A_1/A_o decreases.

Figures 7 and 8 are plots of the length of the extended wire, a/2, as a function of the design angular rate, $\dot{\alpha}_o$, for steel and glass, respectively, again for various values of A_1/A_o. A comparison of Figs. 3, 4, 7, and 8 shows that the increase in length by the cylindrical extension ranges from about 20 per cent at a value of $A_1/A_o = 10^{-1}$ to about 3 per cent for $A_1/A_o = 10^{-6}$.

Constant Cross Section

For comparison purposes, it is of interest to determine the stress relation and possible length of a wire of uniform cross section. This is accomplished as follows: If

$$A(x) = A_o \qquad \left(0 < x < \frac{a_o}{2}\right) \tag{23}$$

then Eq. (10) gives the tension force as

$$T(x) = \rho_o \dot{\alpha}^2 A_o \int_x^{\frac{a_o}{2}} \xi \, d\xi \tag{24}$$

which on integration gives

$$T(x) = \frac{\rho_o \dot{\alpha}^2 A_o}{2} \left[\left(\frac{a_o}{2}\right)^2 - x^2\right] \tag{25}$$

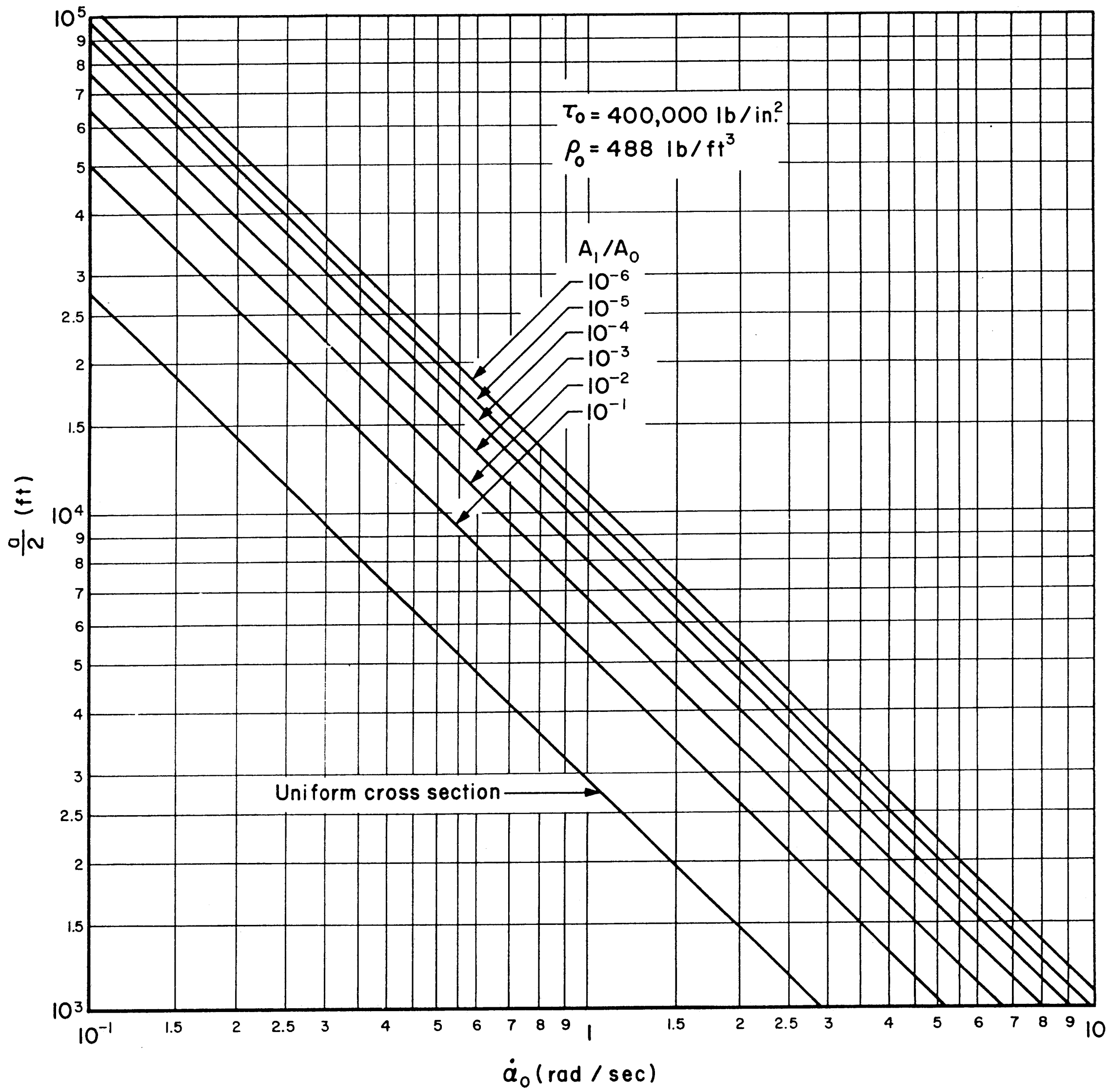

Fig. 7— Length of extended wire vs angular rate

(Steel)

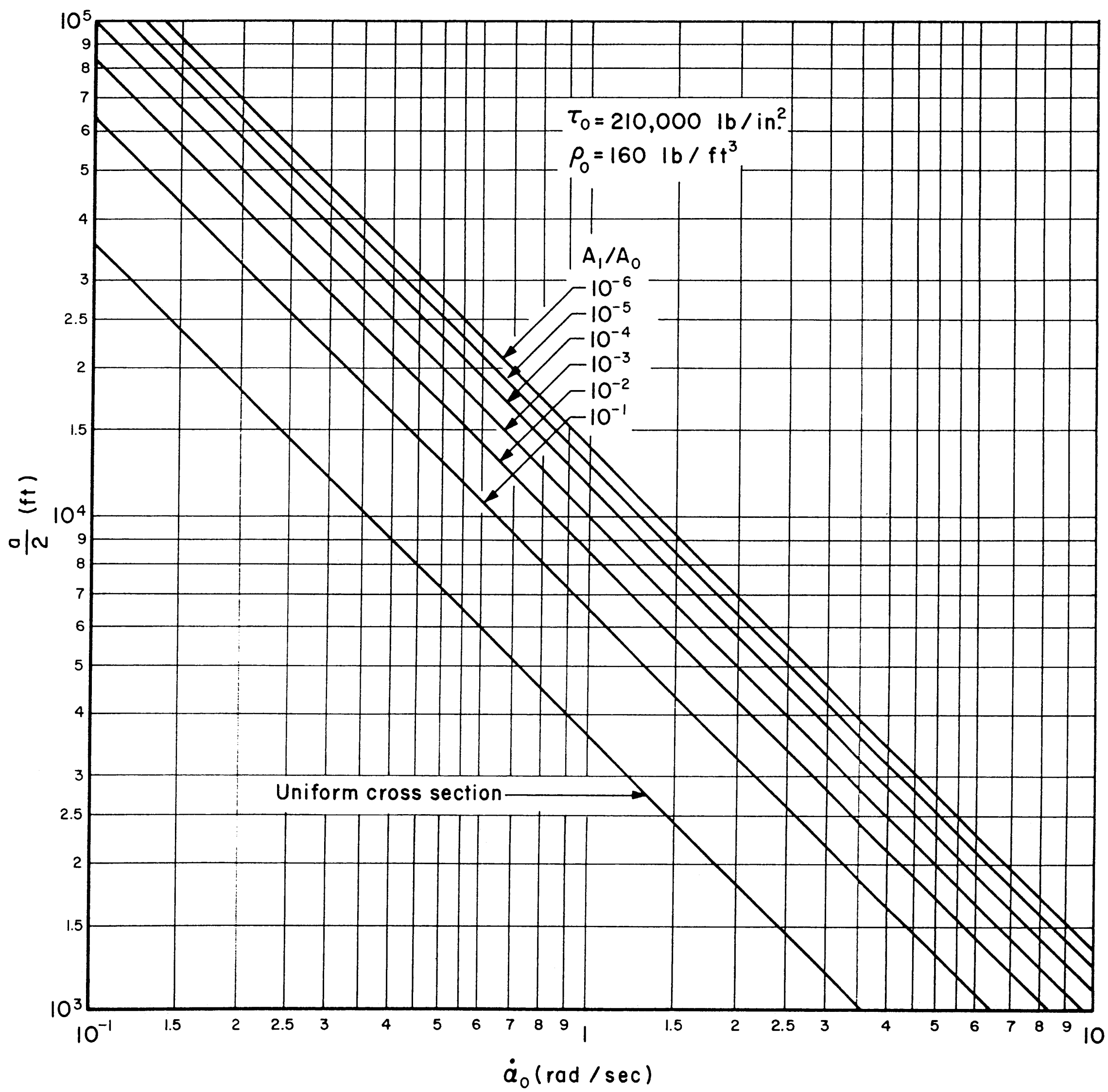

Fig. 8 — Length of extended wire vs angular rate

(Glass)

and for the tensile stress

$$\tau(x) = \frac{\rho_o \dot{\alpha}^2}{2} \left[\left(\frac{a_o}{2} \right)^2 - x^2 \right] \tag{26}$$

At the design rotation rate $\dot{\alpha}_o$, the tensile stress is equal to τ_o at $x = 0$; thus

$$\tau_o = \frac{\rho_o \dot{\alpha}_o^2}{2} \left(\frac{a_o}{2} \right)^2 \tag{27}$$

Combining Eqs. (26) and (27) gives

$$\frac{\tau(x)}{\tau_o} = \frac{\dot{\alpha}^2}{\dot{\alpha}_o^2} \left[1 - \frac{4x^2}{a_o^2} \right] \tag{28}$$

Thus the tensile stress varies parabolically from a maximum of τ_o at $x = 0$ and $\dot{\alpha} = \dot{\alpha}_o$ to zero at $x = a_o/2$, as shown in Fig. 9. A comparison of Fig. 9 with Figs. 2 and 6 shows that the uniform cross-section case makes much less efficient use of the available strength of the material.

The limiting length of the uniform wire is obtained from Eq. (27) as

$$\frac{a_o}{2} = \left[\frac{2\tau_o}{\rho_o \dot{\alpha}_o^2} \right]^{1/2} \tag{29}$$

This relation between $a_o/2$ and $\dot{\alpha}_o$ has been plotted, for comparison purposes, in Figs. 3, 4, 7, and 8. It is seen that in all cases the limiting length for the uniform wire is appreciably less than for the tapered case.

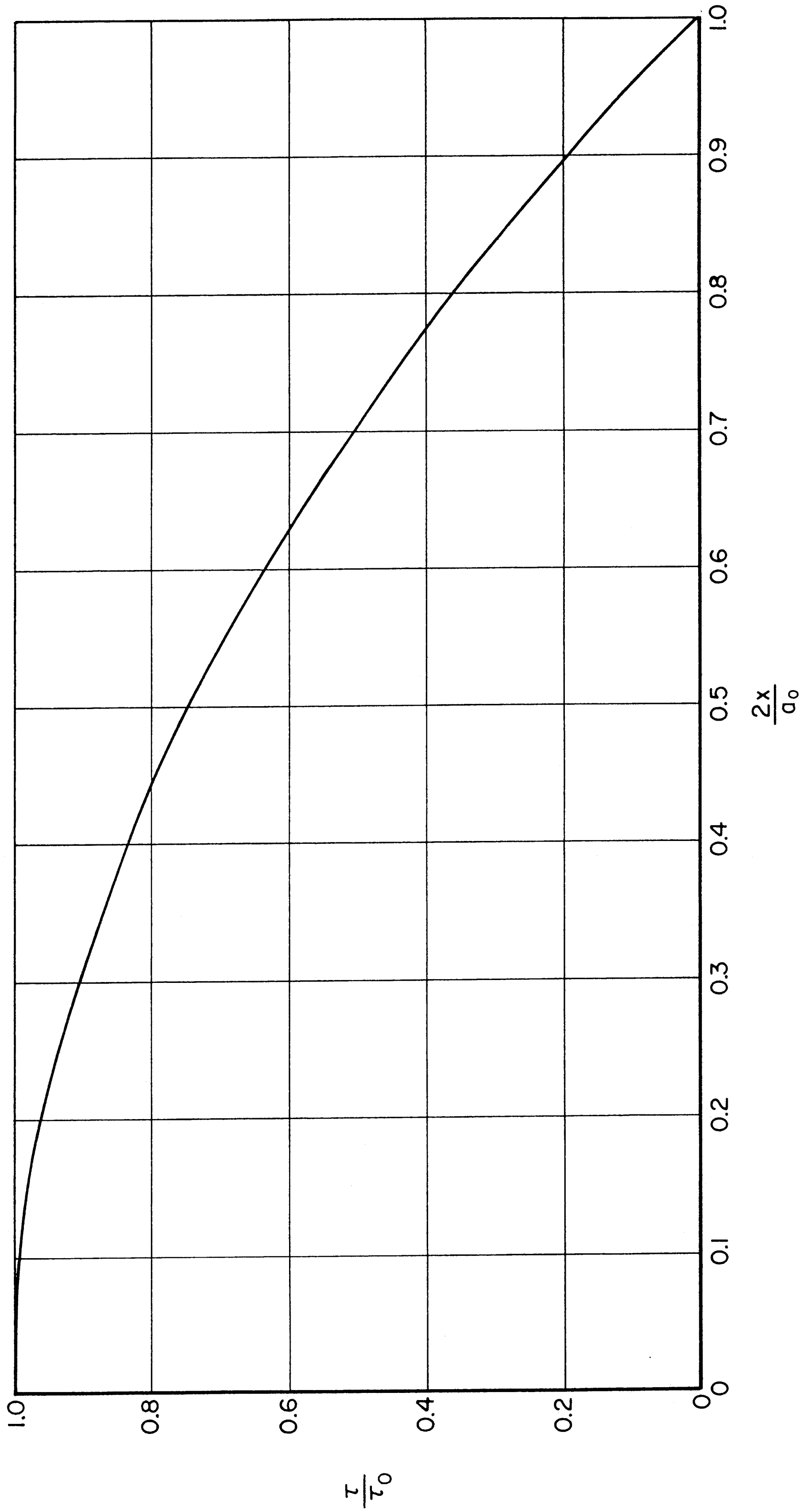

Fig. 9 — Tensile stress in a wire of uniform cross section

III. DISCUSSION OF RESULTS

The foregoing analysis gives a solution to the problem of maximizing the length of a rotating wire antenna under the restrictions that the tensile strength is not exceeded at any point and that the cross-sectional area is not less than some specified value, A_1. The resulting optimum design is made up of two parts. The first segment starts at the axis of rotation with a cross-sectional area, A_o, and tapers in accordance with an error-function variation until the cross-sectional area reaches the minimum allowable value, A_1. The second segment starts at this point and is simply a cylindrical section of cross section A_1 extending to a length such that the tensile stress reaches the tensile strength at some point in the wire.

The resulting tensile stress (shown in Fig. 6) is constant and equal to the tensile strength over the first segment, while in the cylindrical portion the tensile stress varies from τ_o to zero at the outer end. Thus the limitation on tensile stress determines the design of the first segment, while the limitation on the minimum area determines the design of the second segment.

The relation between maximum length, a/2, and the design angular rate, $\dot{\alpha}_o$, shown in Figs. 7 and 8 plotted from Eq. (16), can be characterized by a maximum allowable peripheral velocity of the end of the wire given by

$$V_p = \dot{\alpha}_o \left(\frac{a}{2}\right) = \left[\frac{2\tau_o}{\rho_o}\left(1 + \log \frac{A_o}{A_1}\right)\right]^{1/2} \tag{30}$$

The resulting values of V_p for the cases considered in Figs. 7 and 8 are given in Table 1.

Table 1

MAXIMUM PERIPHERAL VELOCITY

A_1/A_o	V_p (ft/s)	
	Steel	Glass
10^{-1}	4170	5278
10^{-2}	5896	7463
10^{-3}	7222	9141
10^{-4}	8340	10,556
10^{-5}	9324	11,802
10^{-6}	10,212	12,925
Uniform	2749	3478

It should be noted that the velocity V_p is numerically equal to the allowable length for a design rate of rotation of 1 radian per second.

When applying these results to the design of a rotating wire antenna, it would first be necessary to select a material and acceptable values of the maximum and minimum allowable cross-sectional areas, A_o and A_1. These quantities would determine a particular curve in Fig. 7 or 8. The useful part of this curve could then be determined by selecting a minimum allowable length on the basis of acceptable signal reflecting properties, and a minimum rotation rate on the basis of the maximum acceptable time interval between successive reflections. As an example, suppose the antenna is to be made of steel with an area ratio $A_1/A_o = 10^{-1}$. If the minimum acceptable length is given by a = 4000 ft and the minimum angular rate is 0.5 radian per second, then from Fig. 7 it is seen that the resulting range of possible values of a/2 is from 2000 ft to 10,000 ft, or total lengths (including both halves) of 4000 ft to 20,000 ft with corresponding angular rates of 2.52 radians per second and 0.5 radian per second, respectively. On the other hand, for a uniform wire with the same limitations the total

lengths would be 4000 ft to 11,000 ft for rates of 1.38 and 0.5 radians per second, respectively. Thus the process of tapering the wire increases the possible length and the possible angular rate.

IV. CONCLUSIONS

As a result of the preceding analysis, the following conclusions can be stated.

- A variation in cross-sectional area along the wire can be determined which results in maximizing the possible length of the wire for a given angular rate without exceeding the tensile strength and without reducing the cross section below a given minimum value.
- This optimum design, for a given angular rate, permits greater lengths than would be possible for a wire of uniform cross section of the same material.
- This optimum design, for a given length, permits greater angular rates than would be possible with a uniform wire of the same material.
- The possible values of length and angular rate are also increased as the ratio of minimum to maximum cross-sectional area decreases.
- On the basis of numerical examples, it appears that the combinations of length and angular rate which are possible for either steel or glass are in the range of interest in designing a rotating wire antenna which might be extended from a satellite vehicle.